Randomness Inside Nonlinearity

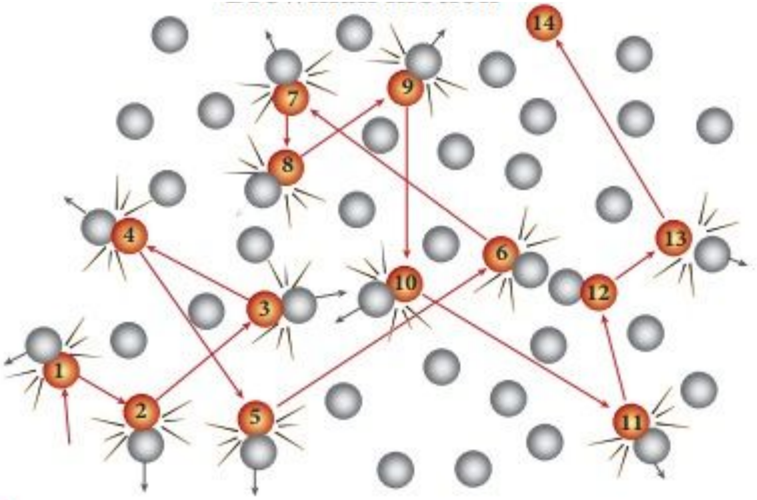
APS March Meeting, March 17 2022

David Sabin-Miller, Northwestern University
Daniel Abrams, Northwestern University

Background: Stochasticity

Many real-world systems exhibit
‘randomness’ in their evolution over time

- Substitutes for unknown/unknowable, or rapidly fluctuating quantity
- 1st: Brownian motion
- Used in finance, ecology, neuroscience, etc



The Standard

- Informally: Langevin Equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) + g(x,t)\eta$$

- Note: random element eta is just multiplied by g(x,t), and that whole random term is just added

- More formally, becomes Itô equation

$$\mathrm{d}x = f(x,t)\mathrm{d}t + g(x,t)\mathrm{d}W$$

- Plenty of theory for analyzing and simulating this

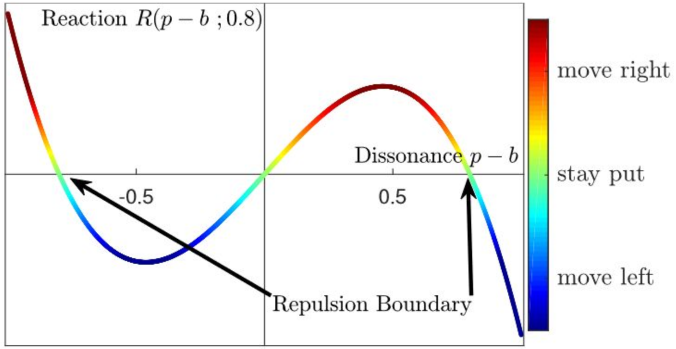
- Ex: $$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^3 + \eta \longrightarrow \mathrm{d}x = -x^3\mathrm{d}t + \mathrm{d}W$$

Our Problem

What if the rapidly fluctuating quantity has a *nonlinear* effect? (not just additive)

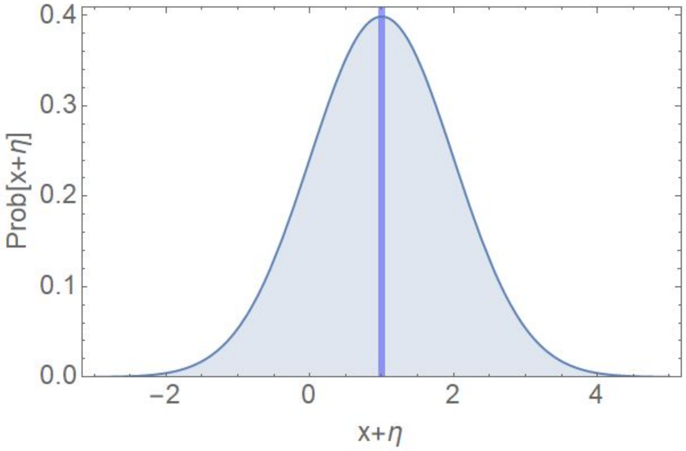
- E.g. environmental influences on individual's political ideology

Ex: $$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x+\eta)^3$$

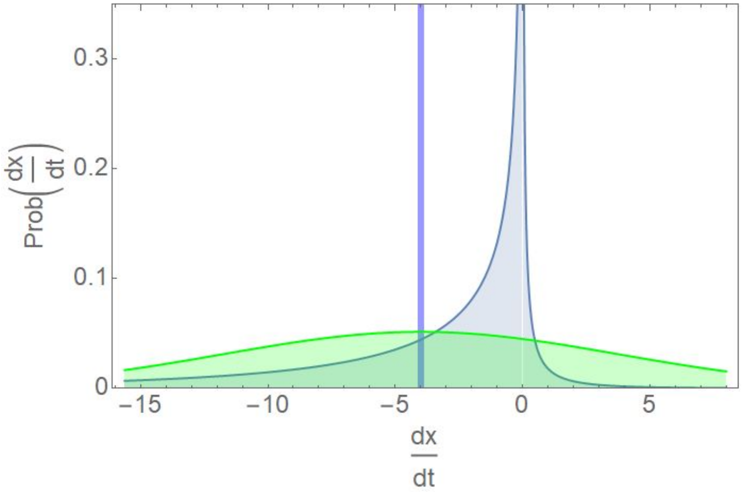


Our (Proposed) Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x+\eta)^3$$



(pictured: x=1)



Only mean and variance-per-time matter for cumulative effect

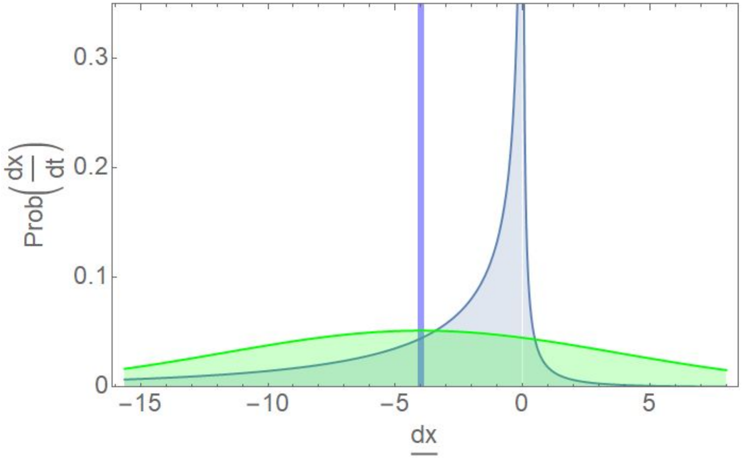
->Use ‘equivalent’ Gaussian

Our (Proposed) Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(x+\eta)^3$$

Only mean and variance-per-time matter for cumulative effect

-> use those for ‘equivalent’ Itô system (effectively, Gaussian which matches)



$$\mathrm{d}x = (-x^3 - 3x)\ \mathrm{d}t + \sqrt{15 + 36x^2 + 9x^4}\ \mathrm{d}W$$

Our (Proposed) Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = R(x, t, \eta) \longrightarrow \mathrm{d}x = F(x, t)\mathrm{d}t + G(x, t)\mathrm{d}W$$

F = mean(R)   G = stdev(R)

Thank you!